ПРОЦЕСИ, ПОВЯЗАНІ ІЗ ЗАСТОСУВАННЯМ КОЛЕКТИВНИХ МОДЕЛЕЙ РИЗИКУ

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PROCESSES RELATED TO THE APPLICATION OF COLLECTIVE RISK MODELS

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Анотація. Є багато книг, досліджень та дослідницьких робіт, які показують деякі основні висновки з математики страхування, не пов'язаного із життям, за допомогою теоретичних страхових операцій. Страхова справа була описана як випадковий і безперервний процес часу. Це дає більш складний погляд на математику страхування і дозволяє застосовувати останні результати теорії стохастичних процесів. Переважаюча думка щодо страхової математики (принаймні серед математиків) полягає в тому, що це досить суха і нудна справа, оскільки інтерпретується лише на мить і насправді не має цікавих структур. Ніхто не повинен сприймати цю точку зору як номінальну, і цікаво працювати з математичними структурами для страхування, не пов'язаного із життям. Можливість отримання абсолютного точного знання існує лише у випадках, пов'язаних із визначенням фізичних параметрів (розмірів, маси, сили тощо) об'єктів оточуючого середовища та за умови використання складних лабораторних методів. У повсякденному житті знання щодо майбутнього, яке знаходиться під впливом значної кількості факторів, що не піддаються вивченню та передбаченню, базуються на основі приблизних оцінок, що і формує поняття невизначеності. Тож поточний сценарій також може бути цікавим для тих, хто не обов'язково хоче провести решту свого життя у страховій компанії. Ці процеси застосовані у статті, щоб представити багато інших областей прикладної теорії ймовірностей, таких як регенеративна теорія, вирівнювання, стохастичні мережі, теорія точкових процесів, застосування рівнянь Пуассона, регенеративні процеси. Де відповідний стохастичний процес називають броунівським рухом, який займається моделюванням вимог, що надходять до страхового бізнесу, який показує, скільки страхової премії слід сплатити, щоб уникнути банкрутства (знищення) страхової компанії, ми маємо на увазі набір контрактів або полісів для подібних ризиків, таких як автострахування, викрадення житла або страхування шкоди від води в односімейних будинках.

Ключові слова: теорія ризику; процеси Пуассона; процес числа претензій; однорідний процес Пуассона; модель Крамера-Лундберга; однорідні; неоднорідні; процеси Маркова; час прибуття.

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Annotation. There are many books, studies and research papers that show some basic findings in the mathematics of non-life insurance through the use of theoretical insurance operations. The insurance business has been described as a random and continuous process of time. This gives a more complex view of insurance mathematics and allows one to apply recent results from the theory of stochastic processes. The prevailing opinion about insurance mathematics (at least among mathematicians) is that it is rather dry and tedious matter because one interprets only momentarily and does not actually have any interesting structures. Nobody should take this view at face value and it is fun to work with mathematical structures for non-life insurance. The possibility of obtaining absolutely accurate knowledge exists only in cases involving the determination of physical parameters of size, mass, force, etc.) of environmental objects and subject to the use of complex laboratory methods. In everyday life, knowledge about the future, which is influenced by a large number of factors that cannot be studied and predicted, is based on approximate estimates, which forms the concept of uncertainty. So the current script can also be interesting for those who don't necessarily want to spend the rest of their lives with an insurance company. These processes lie in this paper to introduce many other fields of applied probability theory, such as regenerative theory, alignment, stochastic networks, point process theory, application such as Poisson, Poisson's compound, and regenerative processes. Where the appropriate stochastic process has been called the Brownian movement, which deals with the modeling of claims that reach the insurance business, which advises on how much insurance premium should be paid to avoid bankruptcy (destruction) of the insurance company by this we mean a set of contracts or policies for similar risks such as auto insurance Certain cars, home theft or water damage insurance in single-family homes.

Key words: risk theory; the Poisson process; claim number process; the homogeneous Poisson process; the Kramer-Lundberg model; homogeneous; Inhomogeneous; Markov processes; arrival times.
Formulas: 77; fig.: 0; tabl.: 0; bibl.: 5.

Introduction. Due to the numerous research and theories in actuarial mathematics having a very good background in measurement theory, probability theory, and stochastic processes, it is natural to present and clarify about non-life insurance based on knowledge of these theories. In particular, the stochastic process theory and applied probability theory (of which insurance mathematics is a part) have made significant progress over the past 50 years, and in this article I highlight presenting some of the fundamental findings in the mathematics of non-life insurance using theoretical processes. It includes the basic model of group risk theory, combining claims volumes and claims arrival times. The claim number process, i.e. the process of calculating claims arrival times, was one of the main things I focused on. Three main operations of claim number are presented: Poisson process, regeneration process, mixed Poisson process, generalized Poisson process or regeneration theory, where these topics are related to reach an understanding of the essay topic, and the total claim amount operations and validity of the basic structure, where random walk is one of the simplest processes. Randomization and in many cases it allows an explicit calculation of the distributions and their properties. And I'll just explain some basic tools like the major revamp on an informal level. Point process theory will be used indirectly in many places, in particular, in the section on Poisson process, and the idea of stochastic scaling will be mentioned.

Literature review. There is a lot of research and studies presented on this topic, for example Ole Hesselager's 1998 notes and exercises for the Basic Course on Non-Life Insurance at the Actuarial Mathematics Laboratory in Copenhagen, and A very interesting person in economic theory and financial mathematics the Russian mathematician Leonid Kantorovich, was specialist in functional analysis In 1938, and was also the book Side Risk by the writer Happy Harry. Applied Stochastic Process Theory.

But through this paper, I will use risk analysis and inference with the applications used through it to find the best results in the field of insurance, special actuarial analysis, and mathematical series study.

Aims. The main objective of this paper is to clarify standard stochastic models for non-life insurance mathematics, to address risks, to clarify the relationship between mathematical chains and actuarial mathematics for non-life insurance, probability theory, random processes, applied stochastic processes, methods of their application, and how to prove theories related to them.

We provide an overview of the definition of risk theory, the Poisson process which is the most common claim number process, the homogeneous Poisson process, the Kramer-Lundberg model where the most common Poisson process corresponds, the Kramer-Lundberg model, the Poisson homogeneous process in insurance mathematics, Markov processes, and the relationship between The Poisson process density function and Markov density, the relationships between the homogeneous and heterogeneous Poisson process, the homogeneous Poisson process as a regeneration process as arrival times, the distribution of arrival times, its applications and their results.

Results. In my research was used risk analysis and its applications to find the results of actuarial mathematical chains, using the Poisson process to find the relationship between the chains processes , and used in the study of random walking, insurance, and meta-analysis, and to find the relationship between homogeneous and heterogeneous chains and their applications.

The Basic Model. Risk theory is a synonym for non-life insurance mathematics, which deals with the modeling of claims that arrive in an insurance business ,and which gives advice on how much premium has to be

charged in order to avoid bankruptcy (ruin) of the insurance company.

One of Lundberg's main contributions is the introduction of a simple mode which is capable of describing the basic dynamics of a homogeneous insurance portfolio.

By this we mean a portfolio of contracts or policies for similar risks such as car insurance for a particular kind of car, insurance against theft in households or insurance against water damage of one-family homes [Thomas, 2006].

There are three assumptions in the model:

Claims happen at the times Ti satisfying ($0 \leq T1 \leq T2 \leq \cdots$) We call them claim arrivals or claim times or claim arrival times or, simply, arrivals.

The *ith* claim arriving at time (Ti)causes the claim size or claim severity (Xi) . The sequence *(Xi)* constitutes an i.i.d sequence of non-negative random variables.

The claim size process *(Xi)* and the claim arrival process *(Ti)* are mutually independent.

The i.i.d property of the claim sizes, Xi , reflects the fact that there is a homogeneous probabilistic structure in the portfolio. The assumption that claim sizes and claim times be independent is very natural from an intuitive point of view. But the independence of claim sizes and claim arrivals also makes the life of the mathematician much easier, i.e., this assumption is made for mathematical convenience and tractability of the model.

Now we can define the claim number process

$$N(t) = \#\{i \geq 1 : Ti \leq t\}, t \geq 0 \qquad (1)$$

i.e., $N = (N(t))$ $t{\geq}0$ is a counting process on $[0,\infty)$: $N(t)$ is the number of the claims which occurred by time t.

The object of main interest from the point of view of an insurance company is the total claim amount process or aggregate claim amount process:

$$S(t) = \sum_{i}^{N(t)} Xi = \sum^{\infty} Xi\, I_{[0,t]}\,(Ti), \qquad t \geq 0 \qquad (2)$$

The process $S = (S(t))t{\geq}0$ is a random partial sum process which refers to the fact that the deterministic index n of the partial

sums $Sn = X1 + \ldots\ldots + Xn$ is replaced by the random variables $N(t)$:

$$S(t) = X1 + \ldots + XN(t) = SN(t), t{\geq}\, 0 \qquad (3)$$

It is also often called a compound (sum) process. We will observe that the total claim amount process S shares various properties with the partial sum process.

For example, asymptotic properties such as the central limit theorem and the strong law of large numbers are analogous for the two processes. [Thomas, 2006], [Yuliya, 2016].

The Poisson Process: we consider the most common claim number process: the Poisson process. It has very desirable theoretical properties. For example, one can derive its finite-dimensional distributions explicitly. The Poisson process has a long tradition in applied probability and stochastic process theory. In his 1903 thesis, Fillip Lundberg already exploited it as a model for the claim number process N. Later on in the 1930s, Harald Cramer, the famous Swedish statistician and probability, extensively developed collective risk theory by using the total claim amount process S with arrivals Ti which are generated by a Poisson process. For historical reasons, but also since it has very attractive Mathematical properties, the Poisson process plays a central role in insurance mathematics [Thomas, 2006].

Below we will give a definition of the Poisson process, and for this purpose we now introduce some notation. For any real-valued function f on $[0,\infty)$ we write :

$$f(s, t] = f(t) - f(s)\, , \, 0 \leq s < t < \infty. \qquad (4)$$

Recall that an integer-valued random variable M is said to have a Poisson distribution with parameter $\lambda > 0$ $(M \sim Pois(\lambda))$ if it has distribution :

$$P\,(M = K) = e^{-\lambda}\, \frac{\lambda^k}{k!}\,,$$
$$K = 0,1,2 \qquad (5)$$

We say that the random variable $M = 0$ a.s. has a *Pois(0)* distribution. Now we are ready to define the Poisson process.

A stochastic process $N = (N(t))t \geq 0$ is said to be a Poisson process if the following conditions hold:

1. The process starts at zero: $N(0) = 0$ a.s.
2. The process has independent increments: for any $ti, i = 0, ... n,$ and $n \geq 1$ such that $0 = t0 < t1 < \cdots < tn,$ the increments $N(ti-1, ti],$ i = 1, . . . , n, are mutually independent.
3. There exists a non-decreasing right-continuous function $\mu : [0,\infty) \rightarrow [0,\infty)$ with $\mu(0) = 0$ such that the increments $N(s, t]$ for $0 \leq s < t < \infty$ have a Poisson distribution $Pois(\mu(s, t])$. We call μ the mean value function of N.
4. The sample paths $(N(t, \omega))t \geq 0$ of the process N are right-continuous for $t \geq 0$ and have limits from the left for $t > 0$. We say that N has sample paths. [Thomas,2006]

We know that a Poisson random variable M has the rare property that

$$\lambda = EM = var(M), \qquad (6)$$

The definition of the Poisson process essentially says that, in order to determine the distribution of the Poisson process N, it suffices to know its mean value function. The mean value function μ can be considered as an inner clock or operational time of the counting process N. Depending on the magnitude of $\mu(s, t]$ in the interval $(s, t], s < t,$ it determines how large the random increment $N(s, t]$ is. [Thomas,2006],[Yuliya,2016]

Since $N(0) = 0$ a.s. and $\mu(0) = 0,$

$$N(t) = N(t) - N(0) = N(0, t] \sim Pois(\mu(0, t]) = Pois(\mu(t)) \qquad (7)$$

We know that the distribution of a stochastic process is determined by its finite-dimensional distributions. The finite-dimensional distributions of a Poisson process have a rather simple structure: for $0 = t0 < t1 < \cdots < tn < \infty,$

$(N(t1),N(t2), . . . , N(tn)) =$
$N(t1),N(t1) + N(t1, t2],N(t1) + N(t1, t2] + N(t2, t3], ... + \sum_{i=0}^{n} N(ti-1, ti] \qquad (8)$

the random variables on the right-hand side is Poisson distributed. The independent increment property makes it easy to work with the finite-dimensional distributions of N: for any integers $ki \geq 0, i = 1, . . . , n,$

$$P(N(t1) = k1 , N(t2) = k1 + k2 , . . ., N(tn) = k1 + \cdots + kn) \qquad (9)$$

$$= P(N(t1) = k1 , N(t1, t2] = k2 , . . . ,N(tn-1, tn] = kn) \qquad (10)$$

$$= e^{-\mu(t1)} \frac{(\mu(t1))^{k1}}{k1!} e^{-\mu(t1,t2]} \frac{(\mu(t1, t2])^{k2}}{k2!} .... \\ e^{-\mu(tn-1,tn]} \frac{(\mu(tn-1,tn])^{kn}}{kn!} \qquad (11)$$

$$= e^{-\mu(tn)} \frac{(\mu(t1))^{k1}}{k1!} e^{-\mu(t1,t2]} \frac{(\mu(t1, t2])^{k2}}{k2!} .... \\ \frac{(\mu(tn-1,tn])^{kn}}{kn!} \qquad (12)$$

The Homogeneous Poisson Process, the Cramer-Lundberg Model

The most popular Poisson process corresponds to the case of a linear mean value function μ:

$$\mu(t) = \lambda t , t \geq 0, \qquad (13)$$

for some $\lambda > 0$. A process with such a mean value function is said to be homogeneous,

Inhomogeneous otherwise. The quantity λ is the intensity or rate of the homogeneous Poisson process. If $\lambda = 1,$ N is called standard homogeneous Poisson process.

More generally, we say that N has an intensity function or rate function λ if μ is absolutely continuous, i.e., for any $s < t$ the increment $\mu(s, t]$ has representation

$$\mu(s,t] = \int_s^t \lambda(y) \, dy , \quad s < t , \qquad (14)$$

for some non-negative measurable function λ. A particular consequence is that μ is a continuous function.

We mentioned that μ can be interpreted as operational time or inner clock of the Poisson

process. If N is homogeneous, time evolves linearly

$\mu(s, t] = \mu(s + h, t + h]$ *for any h > 0 and 0* $\leq s < t < \infty$. (15)

A homogeneous Poisson process with intensity λ has :

1- has sample paths, 2- starts at zero, 3- has independent and stationary increments, 4- $N(t)$ is $Pois(\lambda t)$ distributed for every $t > 0$.

Tationarity of the increments refers to the fact that for any $0 \leq s < t$ and $h > 0$,

$N(s, t] \triangleq N(s + h, t + h] \sim Pois(\lambda (t - s))$, (16)

The Poisson parameter of an increment only depends on the length of the interval ,A process on $[0,\infty)$ called a Levy process. The homogeneous Poisson process is one of the prime examples of Levy processes. [Thomas, 2006], [Yuliya, 2016], [Yuliya & Georgiy 2017].

The Cramer-Lundberg model :

The homogeneous Poisson process plays a major role in insurance mathematics. If we specify the claim number process as a homogeneous Poisson process, the resulting model which combines claim sizes and claim arrivals is called *Cramer-Lundberg model :*

* Claims happen at the arrival times $0 \leq T1 \leq T2 \leq \cdots$ of a homogeneous Poisson process $N(t) = \#\{i \geq 1 : Ti \leq t\}, t \geq 0$.

* The *i*th claim arriving at time *Ti* causes the claim size *Xi*. The sequence *(Xi)*

constitutes an i.i.d sequence of non-negative random variables.

* The sequences *(Ti)* and *(Xi)* are independent. In particular, *N* and *(Xi)* are independent.

The total claim amount process S in the Cramer-Lundberg model is also called a compound Poisson process. [Thomas, 2006], [Yuliya, 2016], [Yuliya & Georgiy 2017].

The Markov Property

Poisson processes constitute one particular class of Markov processes on $[0,\infty)$ with state space $N0 = \{0, 1, \ldots\}$. This is a simple

consequence of the independent increment property. It is left as an exercise to verify the Markov property, i.e., for any $0 = t0 < t1 < \cdots < tn$, and non-decreasing natural numbers

$ki \geq 0, \; i = 1, \ldots, n, \; n \geq 2,$ (17)

$P(N(tn) = kn \mid N(t1) = k1 , \ldots , N(tn−1) = kn−1)$ (18)
$= P(N(tn) = kn \mid N(tn−1) = kn−1)$

Marko process theory does not play a prominent role on modern life insurance mathematics, where Markov models are fundamental.

However, the intensity function of a Poisson process N has a nice interpretation as the intensity function of the Markov process N. Before we make this statement precise, recall that the quantities

$Pk,k+h(s, t) = P(N(t) = k + h \mid N(s) = k) = P(N(t) − N(s) = h)$, (19)

$0 \leq s < t, \quad k,h \in N0$,

are called the transition probabilities of the Markov process N with state space $N0$. Since a.e. path $(N(t, \omega))t \geq 0$ increases, only needs to consider transitions of the Markov process N from k to $k+h$ for $h \geq 0$. The transition probabilities are closely related to the intensities which are given as the limits. [Thomas, 2006], [Yuliya, 2016], [Yuliya & Georgiy 2017].

$: \lambda_{k,k+h}(t)$
$= \lim_{s \to 0} \frac{p_{k,k+h}(t, t+s)}{s}$ (20)

Relation of the intensity function of the Poisson process and its Markov intensities:

Consider a Poisson process $N = (N(t))t \geq 0$ which has a continuous intensity function λ on $[0,\infty)$. *Then, for $k \geq 0$,*

$$\lambda_{k,k+h}(t) = \begin{cases} \lambda(t) & if \;\; h = 1 \\ 0 & if \;\; h > 1 \end{cases}$$

the intensity function $\lambda(t)$ of the Poisson process N is nothing but the intensity of the Markov process N for the transition from state k to state $k + 1$. The intensity function of a Markov process is a quantitative measure of the likelihood that the Markov process N jumps in a small time interval. a Poisson process with continuous intensity function λ has jump sizes larger than 1. Indeed, consider the probability that N has a jump greater than 1 in the interval $(t, t + s]$ for some $t \geq 0, s > 0$:

$P(N(t, t + s] \geq 2) = 1 − P(N(t, t + s] = 0) − P(N(t, t + s] = 1)$ (21)

$= 1 − e^{\mu(t,t+s]} − \mu(t, t + s]\, e^{\mu(t,t+s]}$ (22)

Since λ is continuous,

$\mu(t,\ t+s\ ] = \int_{t}^{t+s} \lambda(y)\, dy = s\lambda(t)\, (1 + o(1)) \to 0$, as $s \downarrow 0$ (23)

Moreover, a Taylor expansion yields for $X \to 0$ that $e^{x} = 1+x + o(x)$.

Thus , we may conclude, as $s \downarrow 0$,

$P(N(t, t + s] \geq 2) = o(\mu(t, t + s]) = o(s)$ (24)

It is easily seen that

$P(N(t, t + s] = 1) = \lambda(t)\, s\, (1 + o(1))$ (25)

Poisson process N with continuous intensity function λ is very unlikely to have jump sizes larger than 1. [Thomas, 2006], [Yuliya, 2016], [Yuliya & Georgiy 2017], [Philip, 2004].

Relations Between the Homogeneous and the Inhomogeneous Poisson Process :

The homogeneous and the inhomogeneous Poisson processes are very closely related: in a deterministic time change transforms a homogeneous Poisson process into an inhomogeneous Poisson process, and vice versa.

Let N be a Poisson process on $[0,\infty)$ with mean value function μ. We start with a standard homogeneous Poisson process $\check{N}$ and define

$\check{N}(t) = \_ N(\mu(t)),\ t \geq 0$ (26)

It is not difficult to see that $\check{N}$ is again a Poisson process on $[0,\infty)$. (Verify

this! Notice that the cadlag property of μ is used to ensure the cadlag property of the sample paths $\check{N}(t, \omega)$.) Since

$\check{N}(t) = E\, \check{N}(t) = E\, \check{N}(\mu(t)) = \mu(t),\ t \geq 0$ (27)

and since the distribution of the Poisson process $\check{N}$ is determined by its mean value function $\hat{\mu}$, it follows that $N \triangleq \check{N}$, where $\triangleq$, refers to equality of the finite-dimensional distributions of the two processes. Hence the processes $\check{N}$ and N are not distinguishable from a probabilistic point of view, in the sense of Kolmogorov's consistency theorem. Moreover, the sample paths of $\check{N}$ are cadlag as required in the definition of the Poisson process.

Now assume that N has a continuous and increasing mean value function μ. This property is satisfied if N has an a.e. positive intensity function λ.

Then the inverse μ^{-1} of μ exists. $\check{N}(t) = N(\mu^{-1}(t))$ is a standard homogeneous Poisson process on $[0,\infty)$ if $lim_{t\to\infty} \mu(t) = \infty$. [Thomas, 2006], [Yuliya, 2016].

The Poisson process under change of time :

Let μ be the mean value function of a Poisson process N and $\check{N}$ be a standard homogeneous Poisson process. Then the following statements hold:

The process $(\check{N}(\mu(t)))t \geq 0$ is Poisson with mean value function μ.

If μ is continuous, increasing and $lim_{t\to\infty} \mu(t) = \infty$ then $(N(\mu-1(t))\ t \geq 0$

is a standard homogeneous Poisson process.

This result, which immediately follows from the definition of a Poisson process, allows one in most cases of practical interest to switch from an inhomogeneous Poisson process to a homogeneous one by a simple time change. In particular,it suggests a straightforward way of simulating sample paths of an inhomogeneous Poisson process N from the paths of a homogeneous Poisson process.

In an insurance context, one will usually be faced with inhomogeneous claim arrival

processes. The above theory allows one to make an "operational time change" to a homogeneous model for which the theory is more accessible. [Yuliya, 2016], [Yuliya & Georgiy 2017], [Thomas, 2006].

The Homogeneous Poisson Process as a Renewal Process :

In this we study the sequence of the arrival times $0 \leq T1 \leq T2 \leq \cdots$ of a homogeneous Poisson process with intensity $\lambda > 0$. It is our aim to find a constructive way for determining the sequence of arrivals, which in turn can be used as an alternative definition of the homogeneous Poisson process. This characterization is useful for studying the path properties of the Poisson process or for simulating sample paths .

We will show that any homogeneous Poisson process with intensity $\lambda > 0$ has representation

$N(t) = \#\{i \geq 1 : Ti \leq t\}, t \geq 0$,

where

$Tn = W1 + \cdots + Wn, \quad n \geq 1$ (28)

and *(Wi)* is an i.i.d exponential *Exp(λ)* sequence. In what follows, it will be convenient to write $T0 = 0$. Since the random walk *(Tn)* with non-negative step sizes *Wn* is also referred to as renewal sequence, a process *N* with representation for a general i.i.d sequence *(Wi)* is called a renewal (counting) process.

The process *N* given with an i.i.d exponential *Exp(λ)* sequence *(Wi)* constitutes a homogeneous Poisson process with intensity $\lambda > 0$.

2. Let *N* be a homogeneous Poisson process with intensity λ and arrival times $0 \leq T1 \leq T2 \leq \cdots$. Then *N* has representation and *(Ti)* has representation for an i.i.d exponential Exp(λ) sequence (Wi).

We start with a renewal sequence *(Tn)* and set $T0 = 0$, for convenience Recall the defining properties of a Poisson process from Definition The property $N(0) = 0$ a.s. follows since $W1 > 0$ a.s. By construction, a path

$(N(t, \omega))t \geq 0$ assumes the value *i* in *[Ti, Ti+1)* and jumps at *Ti+1* to level $i + 1$.

Next we verify that *N(t)* is *Pois(λt)* distributed. The crucial relationship is given by

$\{N(t) = n\} = \{Tn \leq t < Tn+1\}, n \geq 0$ (29)

Since $Tn = W1 + \cdots + Wn$ is the sum of n i.i.d *Exp(λ)* random variables it is a well-known property that *Tn* has a gamma *Γ(n, λ)* distribution10 for $n \geq 1$:

$$P(T_n \leq x) = 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, x \geq 0 \qquad (30)$$

Hence

$$P(N(t) = n) = P(T_n \leq t) - P(T_{n+1} \leq t) = e^{-\lambda x} \frac{(\lambda x)^k}{n!} \qquad (31)$$

This proves the Poisson property of N(t). Now we switch to the independent stationary increment property. We use a direct "brute force" method to prove this property.

Since the case of arbitrarily many increments becomes more involved, we focus on the case of two increments in order to illustrate the method. The general case is analogous but requires some bookkeeping. We focus on the adjacent increments

$N(t) = N(0, t]$ and $N(t, t + h]$ for $t, h > 0$ (32)

We have to show that for any $k, l \in N0$

$q_{k,k+l}(t, t + h) = P(N(t) = k, N(t, t + h] = l)$ (33)

$= P(N(t) = k) P(N(t, t + h] = l)$ (34)

$= P(N(t) = k) P(N(h) = l)$ (35)

$$= e^{-\lambda(t+h)} \frac{(\lambda t)^k (\lambda h)^l}{k! \, l!} \qquad (36)$$

We start with the case $l = 0, k \geq 1$; the case $l = k = 0$ being trivial. We make use of the relation

$\{N(t) = k, N(t, t + h] = l\} = \{N(t) = k, N(t + h) = k + l\}$. (37)

$$q_{k,k+l}(t, t + h) = P(Tk \leq t < Tk+1, Tk \leq t +h < Tk+1) \tag{38}$$

$$= P(Tk \leq t, t +h < Tk + Wk+1) \tag{39}$$

Now we can use the facts that Tk is $\Gamma(k, \lambda)$ distributed with density λk

$x^{k-1} e^{-\lambda x}/(k-1)!$ And $Wk+1$ is $Exp(\lambda)$ distributed with density $\lambda e^{-\lambda x}$

$$q_{k,k+l}(t, t + h) =$$

$$\int_0^t e^{-\lambda z} \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} \int_{t+h-z}^{\infty} \lambda e^{-\lambda x} \, dx dz$$

$$= \int_0^t e^{-\lambda z} \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} e^{-\lambda(t+h-z)} \, dz \tag{41}$$

$$= e^{-\lambda(t+h)} \frac{(\lambda t)^k}{k!}$$

For $l \geq 1$ we use another conditioning argument and : $q_{k,k+l}(t, t + h)$

$$= P(Tk \leq t < Tk+1, Tk+l \leq t +h < Tk+l+1) \tag{43}$$

$$= E[II_{\{Tk \leq t < Tk+1 \leq t+h\}} \tag{44}$$

$$P(Tk+l - Tk+1 \leq t + h - Tk+1 < Tk+l+1 - Tk+1 \mid Tk, Tk+1)] \tag{45}$$

Let $\grave{N}$ be an independent copy of N, i.e., $\grave{N} \triangleq N$. Appealing to the independence of $Tk+1$ and $(Tk+l - Tk+1, Tk+l+1 - Tk+1)$, we see that

$$q_{k,k+l}(t, t + h) \tag{46}$$

$$= E[II_{\{Tk \leq t < Tk+1 \leq t+h\}} P(\grave{N}(t+h - Tk+1) = l-1 / Tk+1)] \tag{47}$$

$$= \int_0^t e^{-\lambda z} \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} \int_{t-z}^{t+h-z} \lambda e^{-\lambda x} P(\grave{N}(t + h - z - x) = l-1) dx dz \tag{48}$$

$$=$$

$$\int_0^t e^{-\lambda z} \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} \int_{t-z}^{t+h-z} \lambda e^{-\lambda x} e^{-\lambda(t+h-z-x)} \frac{(\lambda(t+h-z-x))^{l-1}}{(l-1)!} dx dz \tag{49}$$

$$=$$

$$\int_0^t e^{-\lambda(t+h)} \int_0^t \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} dz \int_0^h \frac{\lambda(\lambda x)^{l-1}}{(l-1)!} dx$$

$$= e^{-\lambda(t+h)} \frac{(\lambda t)^k}{k!} \frac{(\lambda h)^l}{l!} \tag{50}$$

it also follows that $P(N(t) = k, N(t, t + h] = l) = P(N(t) = k) P(N(h) = l)$. If I have enough patience for finitely many increments of N. [Yuliya, 2016], [Yuliya & Georgiy 2017], [Thomas, 2006].

Consider a homogeneous Poisson process with arrival times $0 \leq T1 \leq T2 \leq \cdots$ and intensity $\lambda > 0$. We need to show that there exist i.i.d exponential $Exp(\lambda)$ random variables Wi such that $Tn = W1 + \cdots + Wn$, i.e., we need to show that, for any $0 \leq x1 \leq x2 \leq \cdots .. \leq xn$, $n \geq 1$,

$$P(T1 \leq x1 \ldots Tn \leq xn) \tag{52}$$

$$= p(W1 \leq x1 \ldots W1 + \ldots + Wn \leq xn) \tag{53}$$

$$\int_{w1=0}^{x1} \lambda e^{-\lambda w1} \int_{w2=0}^{x2-w1} \lambda e^{-\lambda w2} \ldots \int_{wn=0}^{xn-w1-\cdots-wn-1} \lambda e^{-\lambda wn} \, dwn \ldots \ldots dw1 \tag{54}$$

The verification of this relation is left as an exercise. Hint: It is useful to exploit the relationship

$$\{T1 \leq x1, \ldots, Tn \leq xn\} = \{N(x1) \geq 1, N(xn) \geq n \tag{55}$$

for $0 \leq x1 \leq \cdots \leq xn, n \geq 1$.

An important consequence of Theorem is that the inter-arrival times

$$Wi = Ti - Ti-1, i \geq 1 \tag{56}$$

of a homogeneous Poisson process with intensity λ are i.i.d $Exp(\lambda)$. In particular, $Ti < Ti+1$ a.s. for $i \geq 1$, i.e., with probability 1 a homogeneous Poisson process does not have jump sizes larger than 1.

Since by the strong law of large numbers Tn/n $a\rightarrow.s.$ $EW1 = \lambda-1 > 0$, we may also conclude that Tn grows roughly like n/λ, and therefore there are no limit points in the sequence (Tn) at any finite instant of time. This means that the values $N(t)$ of a homogeneous Poisson process are finite on any finite time interval $[0, t]$. The Poisson process has many amazing properties. One of them is the following phenomenon which runs in the literature under the name inspection paradox [Thomas, 2006],

The inspection paradox:

Assume that you study claims which arrive in the portfolio according to a homogeneous Poisson process N with intensity λ. We have learned that the inter-arrival times $Wn = Tn - Tn-1$, $n \geq 1$, with $T0 = 0$, constitute an i.i.d $Exp(\lambda)$ sequence. Observe the portfolio at a fixed instant of time t. The last claim arrived at time $TN(t)$ and the next claim will arrive at time $TN(t)+1$. Three questions arise quite naturally:

(1) What is the distribution of $B(t) = t-TN(t)$, i.e., the length of the period $(TN(t), t]$ since the last claim occurred?

(2) What is the distribution of $F(t) = TN(t)+1-t$, i.e., the length of the period $(t, TN(t)+1]$ until the next claim arrives?

(3) What can be said about the joint distribution of $B(t)$ and $F(t)$?

The quantity $B(t)$ is often referred to as *backward recurrence time* or *age*, whereas $F(t)$ is called *forward recurrence time*, *excess life* or *residual life*. Intuitively, since t lies somewhere between two claim arrivals and since the inter-arrival times are i.i.d $Exp(\lambda)$ we would perhaps expect that $P(B(t) \leq x1) < 1 - e^{-\lambda x1}$, $x1 < t$, and $P(F(t) \leq x2) < 1 - e^{-\lambda x2}$, $x2 > 0$.

Howeve these conjectures are not confirmed by calculation of the joint distribution function of $B(t)$ and $F(t)$ for $x1$, $x2 \geq 0$:

$$G_{B(t),F(t)} (x1, x2) = P(B(t) \leq x1 , F(t) \leq x2) . \qquad (57)$$

Since $B(t) \leq t$ a.s. we consider the cases $x1 < t$ and $x1 \geq t$ separately. We observe for $x1 < t$ and $x2 > 0$,

$$\{ B(t) \leq x1 \} = \{ t - x1 \leq T_{N(t)} \leq t \} = \{ N(t - x1, t] \geq 1 \} \qquad (58)$$

$$\{ F(t) \leq x2 \} = \{ t < T_{N(t)+1} \leq t + x2 \} = \{ N(t, t + x2] \geq 1 \} \qquad (59)$$

Hence, by the independent stationary increments of N

$$G_{B(t),F(t)} (x1, x2) = P (N(t - x1, t] \geq 1 , N(t, t + x2] \geq 1) \qquad (60)$$

$$= P (N(t - x1, t] \geq 1) P (N(t, t + x2] \geq 1) \qquad (61)$$

$$= (1 - e^{-\lambda x1}) \cdot (1 - e^{-\lambda x2}) \qquad (62)$$

An analogous calculation for $x1 \geq t$, $x2 \geq 0$

$$G_{B(t),F(t)} (x1, x2) = [(1 - e^{-\lambda x1}) I_{[0,t)}(x1) + I_{[0,\infty)}(x1)] (1 - e^{-\lambda x2})$$

Hence $B(t)$ and $F(t)$ are independent, $F(t)$ is $Exp(\lambda)$ distributed and $B(t)$ has a truncated exponential distribution with a jump at t:

$$P(B(t) \leq x1) = 1 - e^{-\lambda x1} , \ x1 < t , \ and \ P(B(t) = t) = e^{-\lambda t} . \qquad (63)$$

This means in particular that the forward recurrence time $F(t)$ has the same $Exp(\lambda)$ distribution as the inter-arrival times Wi of the Poisson process N.

This property is closely related to the *forgetfulness property* of the exponential distribution: $P (W1 > x+ y \mid W1 > x) = P (W1 > y)$, $x,y \geq 0$,

(Verify the correctness of this relation.) and is also reflected in the independent increment property of the Poisson property. It is interesting to observe that

$$\lim_{t \to \infty} P(B(t) \leq x1) = 1 - e^{-\lambda x1} , \ x1 > 0 \qquad (64)$$

Thus, *in an "asymptotic" sense*, both $B(t)$ and $F(t)$ become independent and are exponentially distributed with parameter λ. [Thomas,2006], [Yuliya,2016]

The Distribution of the Inter-Arrival Times:

By virtue of Proposition, an inhomogeneous Poisson process N with mean value function μ can be interpreted as a time changed standard homogeneous $\grave{N}$:

$$(N(t))t \ge 0 \triangleq (\grave{N}\ (\mu(t)))t \ge 0 \qquad (65)$$

In particular, let ($\widehat{T\iota}$) be the arrival sequence of $\grave{N}$ and μ be increasing and continuous. Then the inverse μ^{-1}exists and

$$N(\grave{t})\_(t) = \#\{\ i \ge 1 : \grave{T}i \le \mu(t)\ \} = \#\{\ i \ge 1 : \mu^{-1}\ (\grave{T}i) \le t\ \},\ t \ge 0\ , \qquad (66)$$

is a representation of N in the sense of identity of the finite-dimensional distributions, i.e., $N \triangleq \grave{N}$. Therefore and by virtue of Theorem the arrival times of an inhomogeneous Poisson process with mean value function μ have representation*:*

$$Tn = \mu^{-1}\ (\grave{T}n)\ ,\quad \grave{T}n\ =\ \grave{W}1\ +\ \cdots\ +\ \grave{W}n, \quad n \ge 1\ ,\ \grave{W}i\ \ i.i.d\ Exp(1). \qquad (67)$$

Joint distribution of arrival/inter-arrival times :

Assume N is a Poisson process on $[0,\infty)$ with a continuous a.e. positive intensity function λ. Then the following statements hold.

The vector of the arrival times *(T1 … Tn)* has density

$$f_{T1.......Tn}\ (x1 ... ., xn) = e^{-\mu(xn)}\ \prod_{i=1}^{n} \lambda(xi)\ I_{\{0<x1<\cdots<xn\}} \qquad (68)$$

The vector of inter-arrival times *(W1, . . .,Wn) = (T1, T2 − T1, . . . , Tn − Tn−1)* has density

$$f_{W1.......Wn}\ (x1 ... ., xn)$$
$$= e^{-\mu(x1+\cdots+\cdots+xn)} \prod_{i=1}^{n} \lambda(x1 + \cdots + \cdots + xi)\ ,\ xi \ge 0\ . \qquad (69)$$

Since the intensity function λ is a.e. positive and continuous, $\mu(t) = \int_0^t \lambda(s)$ ds is increasing and μ^{-1} exists. Moreover, μ is differentiable, and $\grave{\mu}$ *(t) = λ(t)*. We make use of these two facts in what follows:

We start with a standard homogeneous Poisson process. Then its arrivals $\grave{T}n$ have representation $\grave{T}n\ = = \grave{W}1\ +\ \cdots\ +\ \grave{W}n$ for an i.i.d standard exponential sequence $(\grave{W}i)$. The joint density of $(\grave{T}1\ +\ \cdots\ +\ \grave{T}n)$ is obtained from the joint density of $(\grave{W}1\ +\ \cdots\ +\ \grave{W}n)$ via the transformation

$$(y1,....... yn)\ \xrightarrow{S}\ (y1, y1+y2, ...y1,....+........+ yn) \qquad (70)$$

$$(z1,....... zn)\ \xrightarrow{S-1}\ (z1, z2\ -z1... zn - zn\text{-}1) \qquad (71)$$

Note that *det $(\partial S(y)/\partial y) = 1$*. Standard techniques for density transformations yield for *0 < x1 < xn*

$$f_{\tilde{T}_{1........}\tilde{T}_n}\ (x1 ..., xn) = f_{\widetilde{W}_{1........}\widetilde{W}_n}\ (\ x1, x2 - x1, ... xn - xn - 1\ ) = e^{-x_n} \qquad (72)$$

$$e^{-(x_2 - x_1)}\ ....\ e^{-(x_n - x_{n-1})} = e^{-x_n} \qquad =$$

Since μ^{-1} exists we conclude that for *0 < x1 < · · · < xn,*

$$P(\ T1\ \le\ x1\ ....\ Tn \le\ xn\ )\ =\ P(\mu^{-1}(\tilde{T}_1) \le x1, ...\mu^{-1}(\tilde{T}_{n1}) \le xn) \qquad (73)$$

$$=\ P(\tilde{T}_1 \le \mu(x1), ...\tilde{T}_n \le \mu(xn)) \qquad (74)$$

$$= \int_0^{\mu(x1)} ... \int_0^{\mu(xn)} f_{\tilde{T}_{1........}\tilde{T}_n}\ (y1 ... yn)\ dyn ... dy1 \qquad (75)$$

$$= \int_0^{\mu(x1)} \ldots \int_0^{\mu(xn)} e^{-y_n} I_{\{y1 < \cdots yn\}} \, dyn \ldots dy1 \quad (76)$$

Taking partial derivatives with respect to the variables $x1, \ldots, xn$ and noticing that $\dot{\mu}(xi) = \lambda(xi)$, we obtain the desired density.

Relation follows by an application of the above transformations S and S^{-1} from the density of $(T1, \ldots, Tn)$:

$$fW1,\ldots,Wn(w1, \ldots, wn) = fT1,\ldots,Tn \ (w1, w1 + w2, \ldots, w1 + \cdots + wn). \quad (77)$$

we may conclude that the joint density of $W1, \ldots, Wn$ can be written as the product of the densities of the Wi's if and only if $\lambda(\cdot) \equiv \lambda$

for some positive constant λ. This means that only in the case of a homogeneous Poisson process are the inter-arrival times

$W1 \ldots Wn$ independent (and identically distributed). This fact is another property which distinguishes the homogeneous Poisson process within the class of all Poisson processes on $[0, \infty)$.

From previous results of gauge theory, probability theory, and random processes, it is natural to study non-life insurance based on knowledge of these theories. In particular, it has investigated the theory of stochastic processes and the theory of applied probability, and thus it seems appropriate to use these tools for example Martingale and Markov process theory is avoided as much as possible as well as many analytical tools such as Laplace-Stilt transformations, measuring margins and instead, focusing on a more probabilistic understanding. Intuitive risk and lump sum claim operations and its basic random structure. Random walk is one of the simplest stochastic processes and in many cases allows explicit calculations of the distributions and their properties. If one walks this way, then one is essentially walking along the path of regeneration and making the point. However, only some basic tools such as the main theory of regeneration will be explained on an informal level. Point process theory will be used indirectly in many places, in particular, in the section on Poisson's

process, the idea of stochastic measure will be mentioned but it is not necessary, since theoretical arguments can sometimes be replaced by intuitive ones.

Conclusions In this paper, we consider collective risk models risk theory, the Poisson process which is the most common claim number process, and the cautery processes obtained by analyzing risk management data using probability distributions, where the results obtained through different algorithms are compatible with Each other in their basic features, and in this way we were able to compile and direct them that differs from what is common in other research that reflects different views on the method of outputs and analysis of results, as these results allow us to describe the similarities and differences through the relationship between homogeneous and heterogeneous mathematical chains. The results of the analysis also showed the comparison between the Poisson process density function and Markov intensity, and the relationships between the homogeneous and heterogeneous Poisson process, and the homogeneous Poisson process as the arrival time distribution process, its applications and results.

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